Optimizing polarimetric parameter estimation in the presence of detector and shot noise

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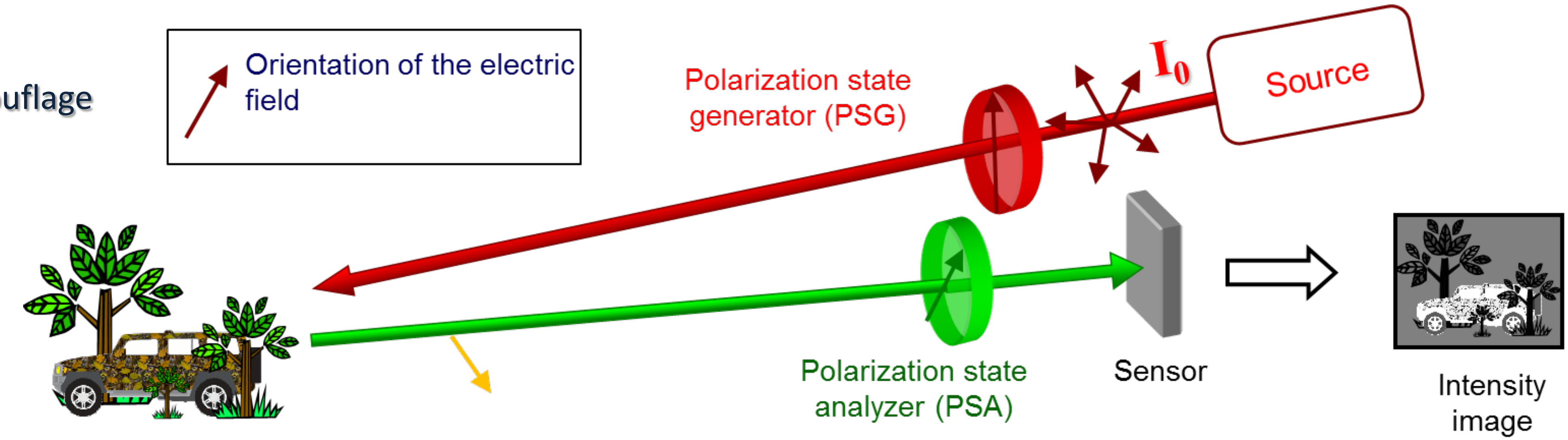
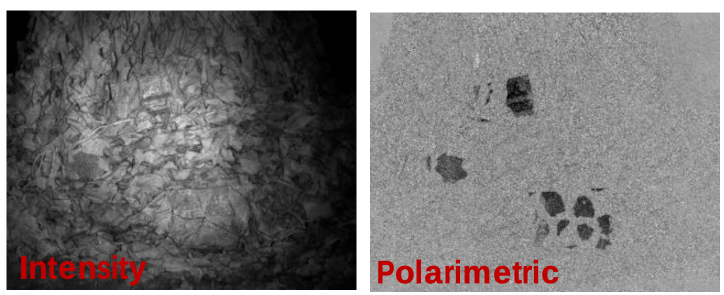
Why a polarimetric imager ?

Décamouflage

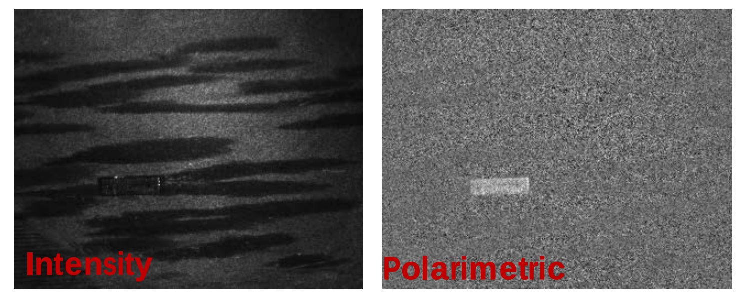


Many applications

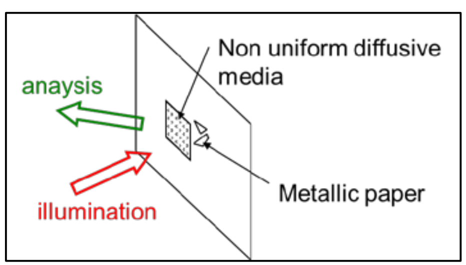
Decamouflage

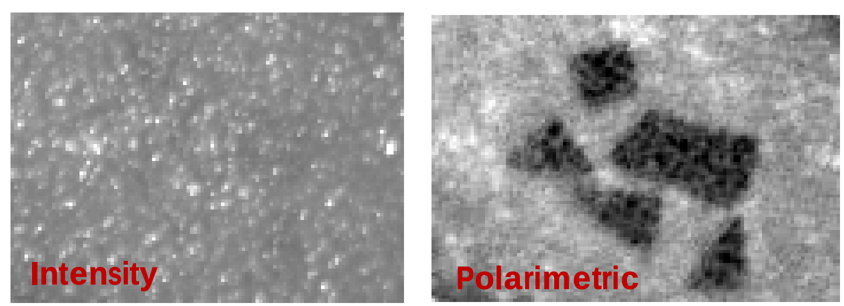


Detection of hazardous objects



Imaging through turbid media



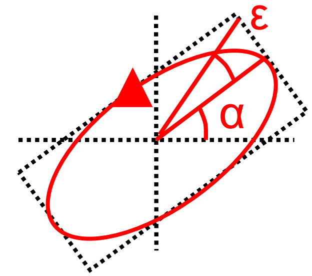


Polarization of light

- **Polarization = time variation of electric field vector**

  - Fully polarized: deterministic trajectory → ellipse in general case

    ↳ Characterized by intensity **I**, **azimuth α** and **ellipticity ε**



  - Totally unpolarized: totally random trajectory

  - Partially polarized : superposition of a totally polarized state and a totally depolarized one.

    ↳ Described by **Stokes vector :**

$$\mathbf{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = I \begin{bmatrix} 1 \\ P\cos(2\alpha)\cos(2\epsilon) \\ P\sin(2\alpha)\cos(2\epsilon) \\ P\sin(2\epsilon) \end{bmatrix}$$

Stokes vector - Poincaré sphere

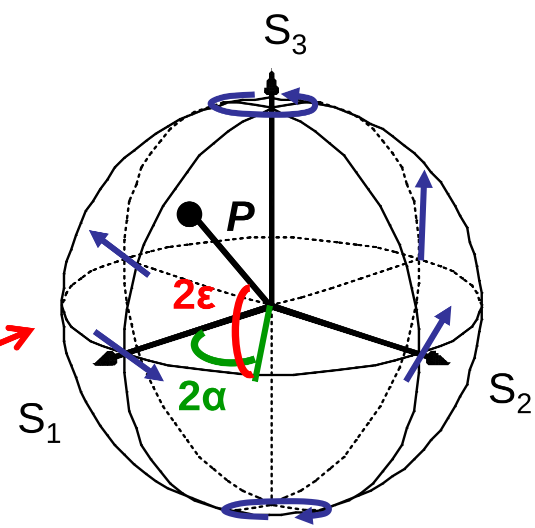
Graphical representation of polarisation state

Degree of polarization (DOP)

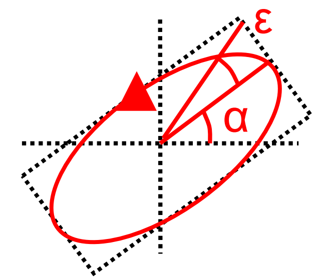
$$\mathbf{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = I \begin{bmatrix} 1 \\ P\cos(2\alpha)\cos(2\epsilon) \\ P\sin(2\alpha)\cos(2\epsilon) \\ P\sin(2\epsilon) \end{bmatrix}$$

Stokes vector

Intensity

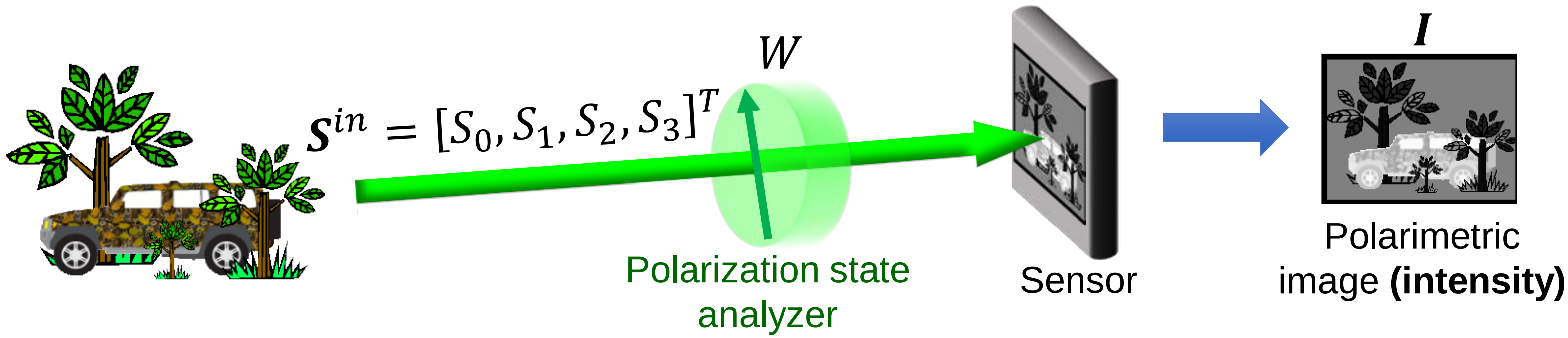


Poincaré sphere



Polarization ellipse

Polarization state estimation



Measurement matrix

- One has: $\mathbf{I} = W\mathbf{S}^{in}$ $\qquad \mathbf{I} = \begin{bmatrix} i_1 \\ \vdots \\ i_N \end{bmatrix} \qquad W = \frac{1}{2}\begin{bmatrix} \mathbf{T}^T(\theta_1) \\ \vdots \\ \mathbf{T}^T(\theta_N) \end{bmatrix}$

⇨ We want to estimate $\mathbf{S}^{in}$ from measurement of $\mathbf{I}$.

- o If N=4 and W invertible : $\widehat{\mathbf{S}} = W^{-1}\mathbf{I}$
- o If N>4 (pseudo-inverse): $\widehat{\mathbf{S}} = W^{+}\mathbf{I} \qquad W^{+} = (W^TW)^{-1}W^T$

Optimizing Stokes vector measurements

- In practice, **measurements are perturbed by noise** (assumed additive and white) :

$$\mathbf{I} = W\mathbf{S}^{in} + \mathbf{B} \quad \Rightarrow \quad \widehat{\mathbf{S}} = \mathbf{S}^{in} + W^{+}\mathbf{B}$$

$$\Gamma_{\mathbf{B}} = \langle \mathbf{B}\mathbf{B}^T \rangle = \sigma^2 \mathrm{Id}$$

- What is the **optimal** matrix W?
- What happens if W is **not optimal**?

Optimizing Stokes vector measurements

- **Reasonable criterion** : minimize sum of variances of $\widehat{\mathbf{S}}$ coefficients

« Equally Weighted Variance »

$$EWV = \sum_{k=1}^{4} VAR[\widehat{S}_k] = \mathrm{tr}[\Gamma_{\widehat{\mathbf{S}}}]$$

$$\widehat{\mathbf{S}} = \mathbf{S}^{in} + W^{+}\mathbf{B}$$

$$\Gamma_{\mathbf{B}} = \langle \mathbf{B}\mathbf{B}^T \rangle = \sigma^2 \mathrm{Id}$$

where:

$$\Gamma_{\widehat{\mathbf{S}}} = \left\langle (\widehat{\mathbf{S}} - \mathbf{S}^{in})(\widehat{\mathbf{S}} - \mathbf{S}^{in})^T \right\rangle = W^{+}\Gamma_{\mathbf{B}}(W^{+})^T$$

- This criterion can be written in the following form:

$$EWV = \mathrm{tr}[W^{+}\Gamma_{\mathbf{B}}(W^{+})^T] = \sigma^2 \mathrm{tr}[(W^T W)^{-1}]$$

- The objective is thus to find the matrix W (or, which is equivalent, the projection vectors **T**(θ_n)) that **minimises**:

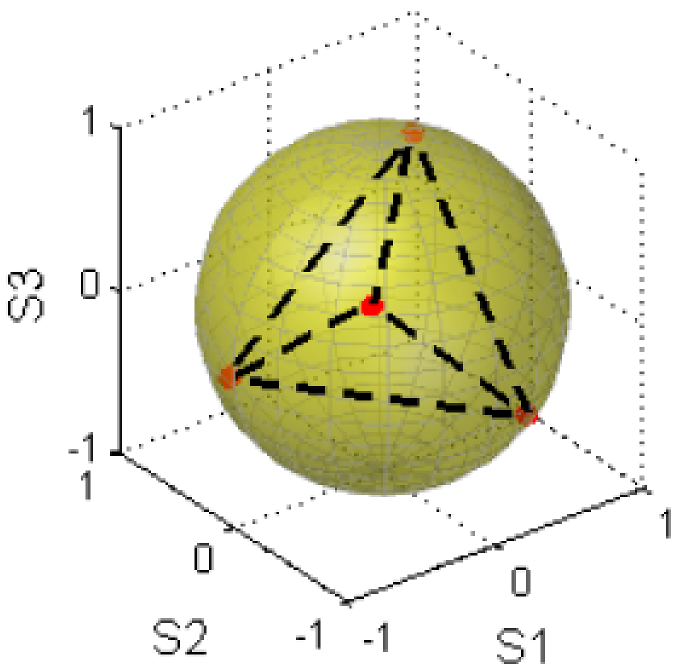
$$\mathrm{tr}[(W^T W)^{-1}]$$

Optimizing Stokes vector measurements

$$\min_{W} \left\{ \mathrm{tr}[(W^T W)^{-1}] \right\}$$

$$W = \frac{1}{2} \begin{bmatrix} \mathbf{T}^T(\theta_1) \\ \vdots \\ \mathbf{T}^T(\theta_N) \end{bmatrix}$$

- **If N=4** : the optimal solution consists in choosing vectors $\mathbf{T}(\theta_n)$ that define a **regular tetrahedron** on the Poincaré sphere.



- In this case, the covariance matrix of the estimator is :

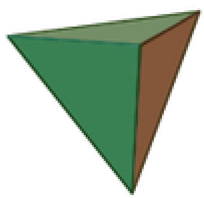
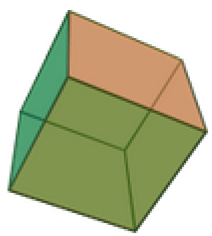
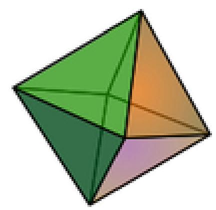
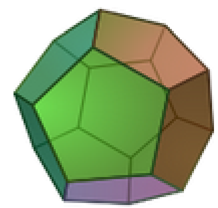
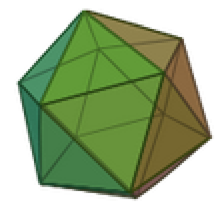
$$\Gamma_{\hat{\mathbf{S}}} = \sigma^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Azzam et al., J. Opt. Soc. Am. A, 5, 681 (1988)
Sabatke et al., Opt. Lett., 25, 802 (2000)

Optimizing Stokes vector measurements

- **If N>4 :** the optimal solution consists in choosing vectors $T(\theta_n)$ that define a **spherical-2 design** on the Poincaré sphere.

- The **platonic solids** are spherical-2 designs

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
				
N=4	**N=8**	**N=6**	**N=20**	**N=12**

- **The covariance matrix** of the estimator is:

$$\Gamma_{\hat{\mathbf{s}}} = \frac{4}{N}\sigma^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

- **The EWV** is:

$$EWV_{opt} = \frac{40\sigma^2}{N}$$

Foreman et al., Phys. Rev. Lett. (2015)

Foreman, Goudail, Opt. Eng. (2019)

Optimization in the presence of photon noise

- In practice, for sufficient level of light, **photon noise** dominates.

What is the optimal configuration in this case ?

- In this case, measured intensities i_n are **Poisson random variables** whose mean (and variance) is equal to : $\frac{1}{2}\mathbf{T}^T(\theta_n)\mathbf{S}^{in}$

  They are **statistically independent**.

- One has :

$$\widehat{\mathbf{S}} = W^+\mathbf{I} \quad \Rightarrow \quad \Gamma_{\widehat{\mathbf{S}}} = W^+\Gamma_{\mathbf{I}}(W^+)^T$$

with:

$$[\Gamma_{\mathbf{I}}]_{mn} = \begin{cases} \frac{1}{2}\mathbf{T}^T(\theta_n)\mathbf{S}^{in} & \text{if} \quad m = n \\ 0 & \text{otherwise} \end{cases}$$

Optimization in the presence of photon noise

- The criterion to optimize **is the same** as in the case of additive Gaussian noise :

$$C = \sum_{k=1}^{4} VAR[\widehat{S}_k] = \mathrm{tr}[\Gamma_{\widehat{\mathbf{S}}}]$$

- But now, **it depends on the measured vector $\mathbf{S}^{in}$** : there is one optimal matrix W for each measured vector …

- To solve this problem, one uses a **« minimax »** approach:

$$\min_{W} \left\{ \max_{\mathbf{S}^{in}} \left[\mathrm{tr}\left(\Gamma_{\widehat{\mathbf{S}}}\right) \right] \right\}$$

- **This optimization problem can be solved:** the optimal W matrix is the same as in the case of additive noise, that is, the vectors $T(\theta_n)$ **form a spherical-2 design** on Poincaré sphere.

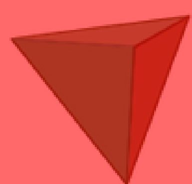
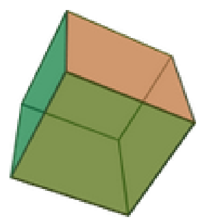
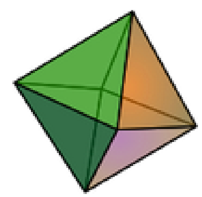
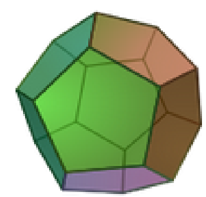
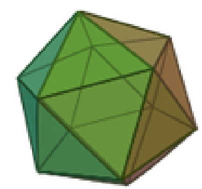
Goudail, Opt. Lett., 2009.

Goudail, Opt. Lett., 2016.

Optimization in the presence of photon noise

- However, the **variances** of each component of Stokes vector may depend on the measured state $\mathbf{S}^{in}$, although their sum is independent.

- They are independent of $\mathbf{S}^{in}$ if the vectors $T(\theta_n)$ **form a spherical-3 design** on Poincaré sphere.

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
				
N=4	**N=8**	**N=6**	**N=20**	**N=12**

- In this case, the **variances of each element** of the estimator are similar as in the presence of additive noise:

$$VAR[S] = \frac{2}{N} S_0^{in} \begin{bmatrix} 1 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

Goudail, Opt. Lett., 2009.

Goudail, Opt. Lett., 2016.

Summary

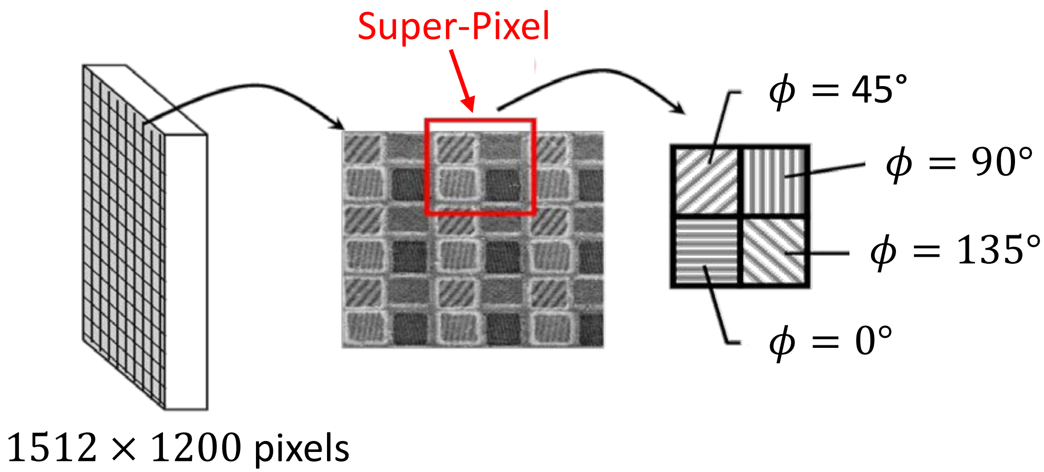
- **Minimization of estimation variance** of Stokes vector in the presence of **additive** and **photon** noise.
  - The optimal configurations are the same in both cases: **spherical-2 designs** on the Poincaré sphere
  - In the presence of **photon noise**, in order to « equalize » the variances, one has to choose **spherical-3 designs**

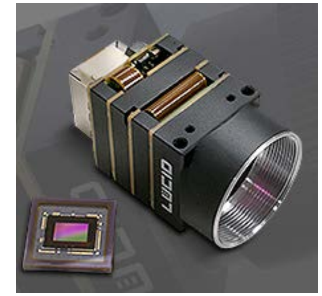
What happens

- if the measurement matrix **is not optimal ?**
- if the **parameters of interest** are not the Stokes vector ?

Division of focal plane polarimetric camera

- Micropolarizer grid on the sensor:





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- Measure of 4 light intensities in one « super-pixel »

$$\boldsymbol{I} = [i_{0°}, i_{45°}, i_{90°}, i_{135°}]$$

$$W = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Estimation of the **linear** Stokes vector: $\boldsymbol{S} = [S_0, S_1, S_2]^T = W^+\boldsymbol{I}$

Defects of the micropolarizer grid

$$W_{\text{ideal}} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

- Orientation $= 0°, 45°, 90°, 135°$
- Diattenuation $= 1$
- Transmission $= 1$

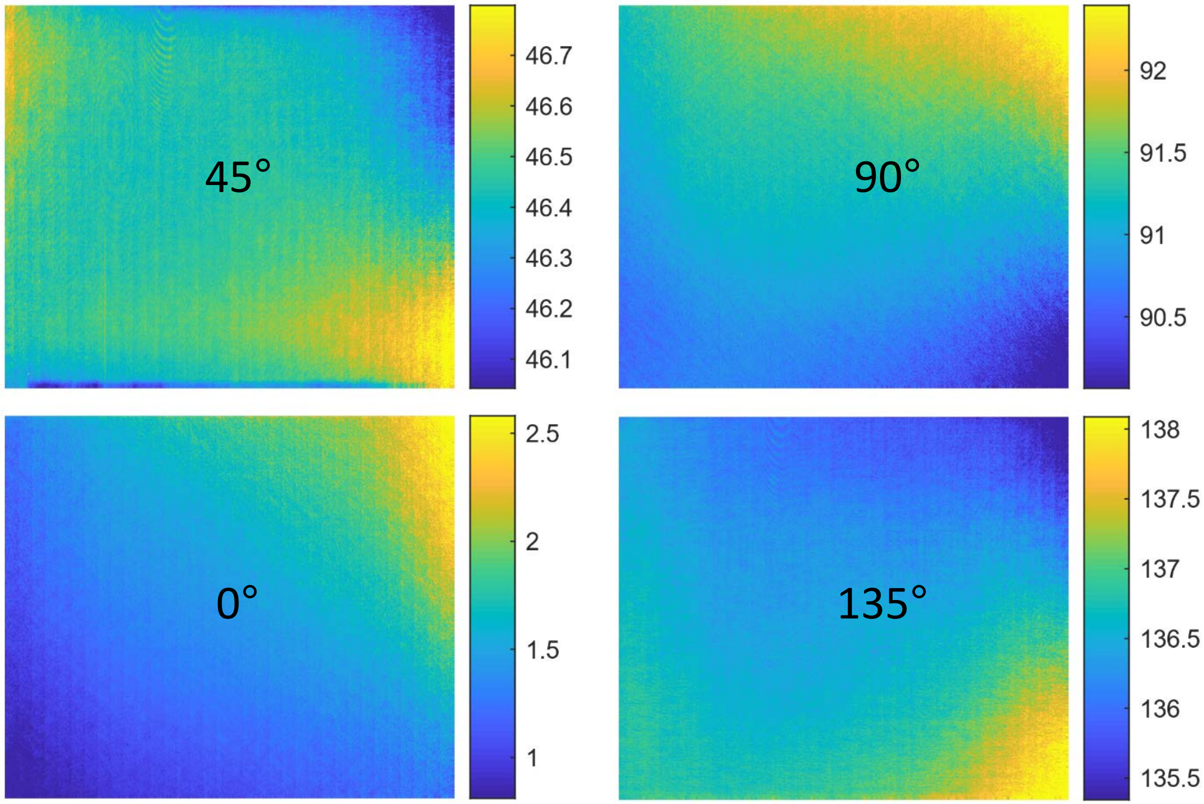
- « Polarimetric » calibration of the camera
  - ➢ Measure the caracteristics of the sensor and of the micropolarizer grid

Example of the measurement matrix W of a real « super-pixel »:

$$W_{\text{real}} = \frac{1}{2}\begin{bmatrix} 0.95 & 0.81 & -0.01 \\ 1.06 & 0.10 & 0.88 \\ 0.97 & -0.78 & 0.08 \\ 0.89 & -0.01 & -0.68 \end{bmatrix}$$

Defects of the micropolarizer grid

Orientation maps of the micro-polarizers



What is the impact of these micropolarizer grid defects on the **estimation performance** of polarimetric parameters (**S**, DOP, AOP)?

Roussel et al., Opt. Express (2018)

Estimation error of the polarimetric parameters

- Estimation of $\boldsymbol{S}$ in the presence additive and Poisson noise

$$\Gamma_{\hat{S}}^{total} = \Gamma^{add} + \Gamma^{poi}$$

Additive noise	Poisson noise
$\boldsymbol{I} = W_{real}\boldsymbol{S} + b$	$\langle \boldsymbol{I} \rangle = \mathrm{VAR}[\boldsymbol{I}] = W_{real}\boldsymbol{S}$
$\Gamma_{\hat{S}}^{add}(i,j) = \sigma_a^2 \delta_{ij}$	$\Gamma_{\hat{S}}^{poi}(i,j) = \sum_{k=0}^{2} S_k \gamma_{ij}^k$

1/g: number of photo-electron per digital level

with

$\delta_{ij} = g^2[(W^T W)^{-1}]_{ij}$ et $\gamma_{ij}^k = g \sum_{l=1}^{4} W_{il}^+ W_{jl}^+ W_{lk}$, $\forall (k,i,j) \in [0,2]^3$

- The linear Stokes vector is estimated with the **calibrated** matrix : $\hat{\boldsymbol{S}} = W_{\mathrm{real}}^+ \boldsymbol{I}$

What is the impact of the non-ideality of W_{real} on **estimation precision** of polarimetric parameters?

Roussel et al., Opt. Express (2018)

Estimation of the linear Stokes vector

Equally Weighted Variance : $\mathrm{EWV} = \mathrm{trace}\left(\Gamma^{\hat{S}}\right) = \mathrm{VAR}(S_0) + \mathrm{VAR}(S_1) + \mathrm{VAR}(S_2)$

Ideal super-pixel

$$\mathrm{EWV}_{\mathrm{ideal}} = 5\left(\sigma_a^2 + \frac{S_0}{2}\right)$$

Depends only on σ_a and S_0

Real super-pixel

Angle of polarization (AOP)

$$\mathrm{EWV}_{real} = \sigma_a^2 \sum_{i=0}^{2} \delta_{ii} + S_0\beta_0\{1 + C\cos[2(\alpha - \theta)]\}$$

- Depends on W
- Depends on α

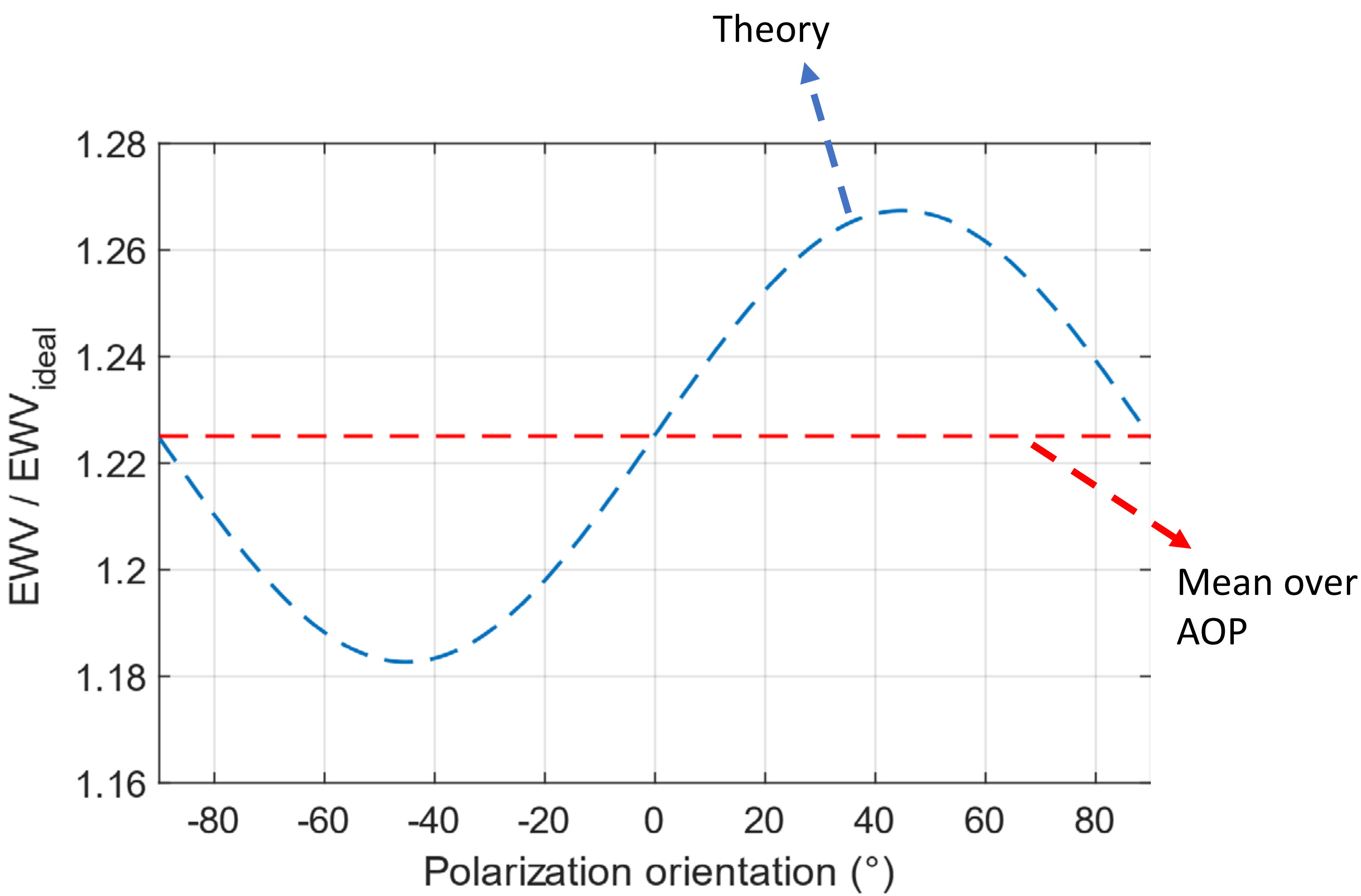
$$\delta_{ij} = g^2[(W^T W)^{-1}]_{ij} \qquad \gamma_{ij}^k = g\sum_{l=1}^{4} W_{il}^{\dagger} W_{jl}^{\dagger} W_{lk}$$

$$\beta_k = \sum_{i=0}^{2} \gamma_{ii}^k$$

$$\theta = \frac{1}{2} arctan\left[\frac{\beta_2}{\beta_1}\right] \qquad C = P\frac{\sqrt{\beta_1^2 + \beta_2^2}}{\beta_0}$$

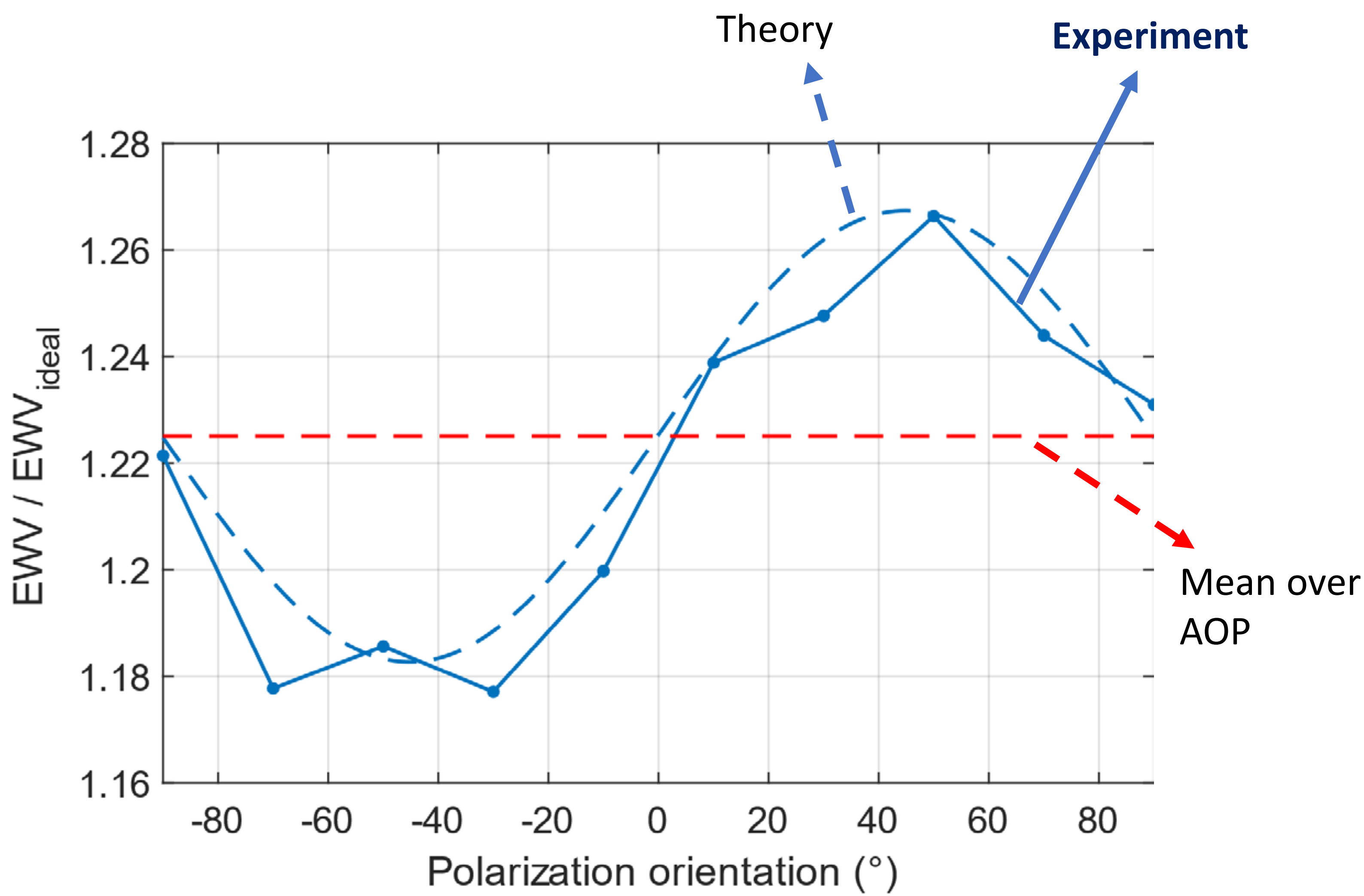
Roussel et al., Opt. Express (2018)

Estimation of the linear Stokes vector



Roussel et al., Opt. Express (2018)

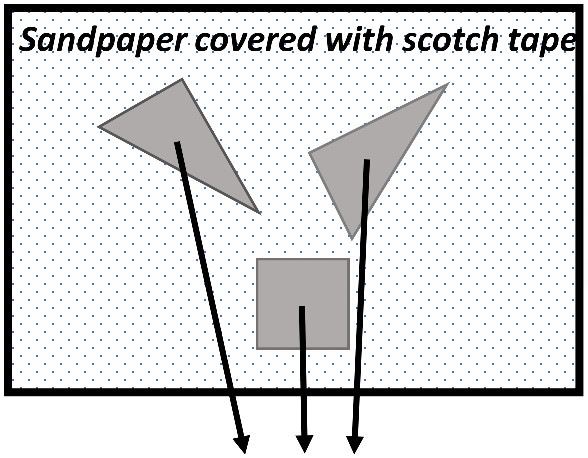
Estimation of the linear Stokes vector



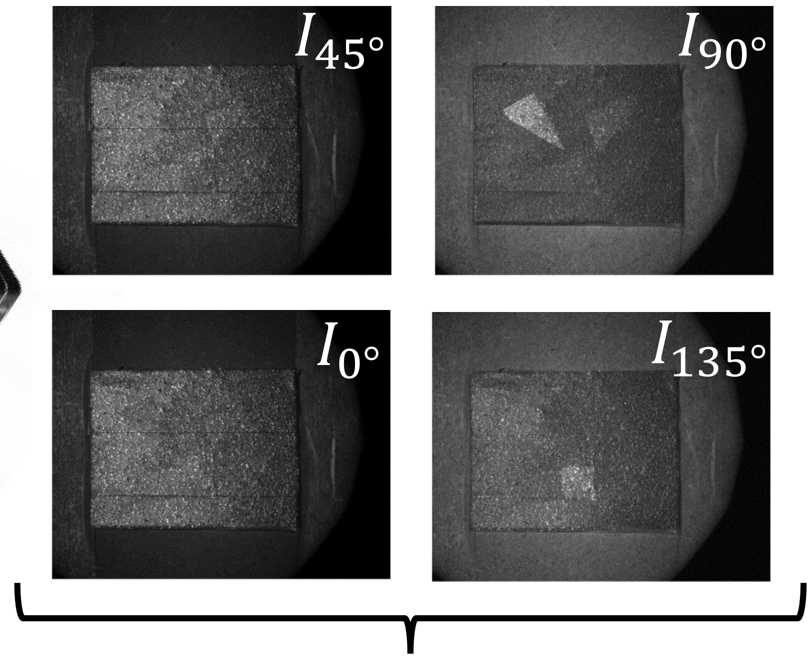
Roussel et al., Opt. Express (2018)

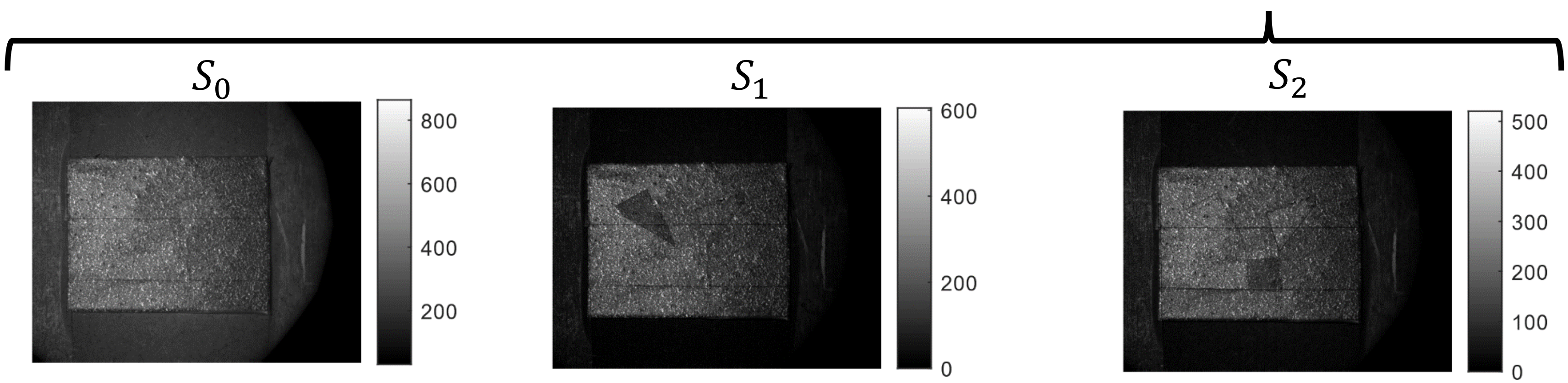
Example of polarimetric image

Scene with birefringence contrast



Three different types of scotch tape





Estimation of polarimetric parameters

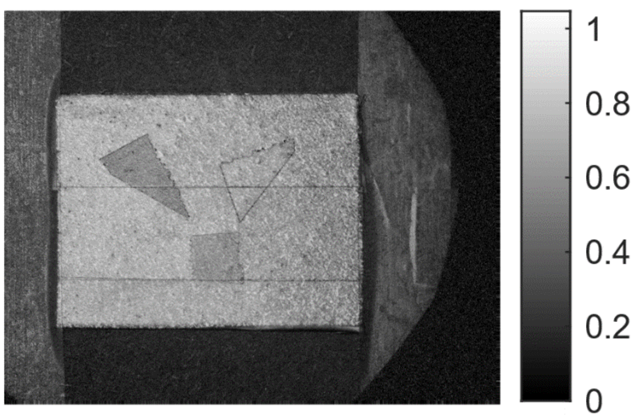
In many applications, the final product is
not the Stokes vector but **other polarimetric parameters**:

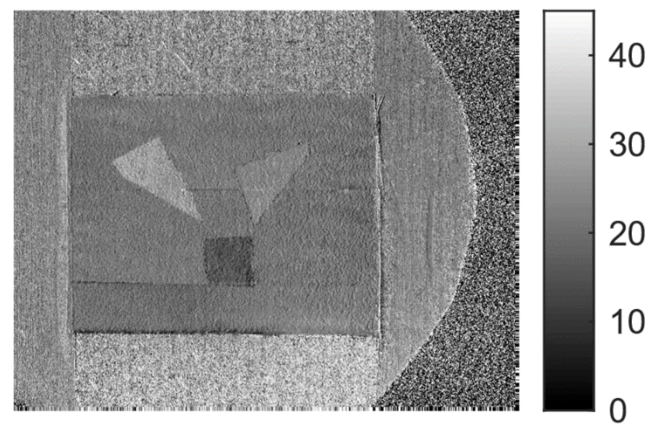
Degree of linear polarization (DOLP)

$$P = \sqrt{\frac{S_1^2 + S_2^2}{S_0}}$$

Angle of polarization (AOP)

$$\alpha = \frac{1}{2}\arctan\left[\frac{S_2}{S_1}\right]$$



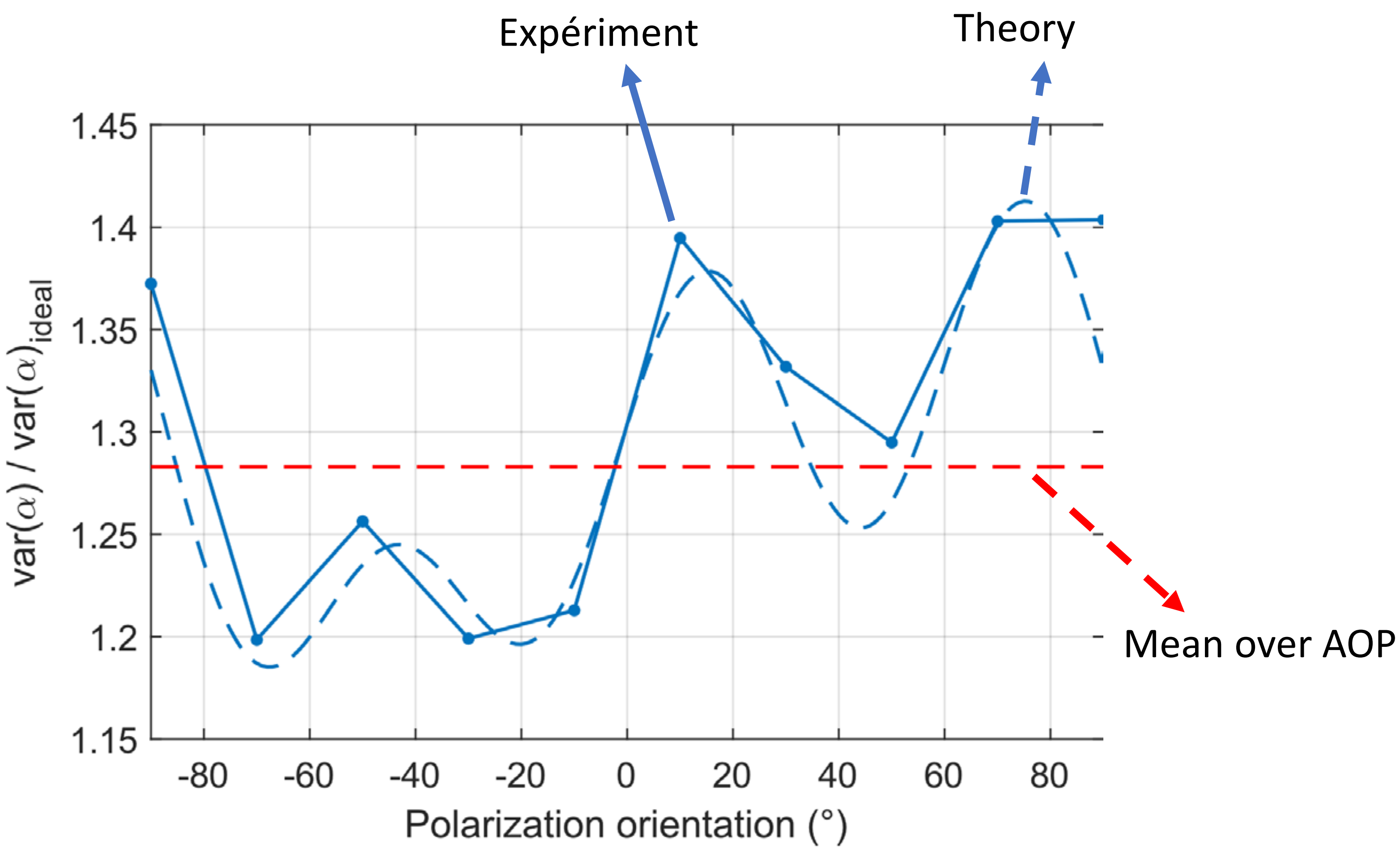


Variance of AOP

Ideal super-pixel	Real super-pixel
$$\mathrm{VAR}[\hat{\alpha}]_{\mathrm{ideal}} = \frac{1}{2P^2}\left(\frac{\sigma_a^2}{S_0^2} + \frac{1}{2S_0}\right)$$	$$\mathrm{VAR}[\hat{\alpha}] = \frac{\sigma_a^2}{4P^2S_0^2}\{\delta_{11}s^2 + \delta_{22}c^2 - 2\delta_{12}cs\} + \frac{1}{4P^2S_0}\{\gamma_{11}^0 s^2 + \gamma_{22}^0 c^2 - 2\gamma_{22}^2 cs + Pc^2[(\gamma_{22}^2 - 2\gamma_{12}^1)s + \gamma_{22}^1 c] + Ps^2[(\gamma_{11}^1 - 2\gamma_{12}^2)c + \gamma_{11}^2 s]\}.$$ with $c = \cos(2\alpha)$ and $s = \sin(2\alpha)$, $\delta_{ij} = g^2[(W^TW)^{-1}]_{ij}$, $\gamma_{ij}^k = g\sum_{l=1}^4 W_{il}^+ W_{jl}^+ W_{lk}$, $\forall(k,i,j) \in [0,2]^3$. • Depends on W • Depends on α

Roussel et al., Opt. Express (2018)

Variance of AOP



Roussel et al., Opt. Express (2018)

Variance of DOP

Ideal super-pixel

$$\mathrm{VAR}[\hat{P}]_{\mathrm{ideal}} = \frac{\sigma_a^2}{S_0^2}[2+P^2] + \frac{1}{2S_0}[2-P^2]$$

Real super-pixel

$$\mathrm{VAR}[\hat{P}] = \frac{\sigma_a^2}{S_0^2}\{P^2\delta_{00} - 2P(\delta_{01}c + \delta_{02}s) + 2\delta_{12}cs + \delta_{11}c^2 + \delta_{22}s^2\} +$$
$$\frac{1}{S_0}\{P^3(\gamma_{00}^1 c + \gamma_{00}^2 s) + P^2[\gamma_{00}^0 - 2\gamma_{01}^1 c^2 - 2\gamma_{02}^2 s^2 - 2(\gamma_{01}^2 + \gamma_{02}^1)cs] +$$
$$P[(\gamma_{11}^2 + 2\gamma_{12}^1)c^2 s + (\gamma_{22}^1 + 2\gamma_{12}^2)cs^2 + \gamma_{11}^1 c^3 + \gamma_{22}^2 s^3 - 2\gamma_{01}^0 c - 2\gamma_{02}^0 s] +$$
$$[\gamma_{11}^0 c^2 + \gamma_{22}^0 s^2 + 2\gamma_{12}^0 cs]\}.$$

- Depends on W
- Depends on α

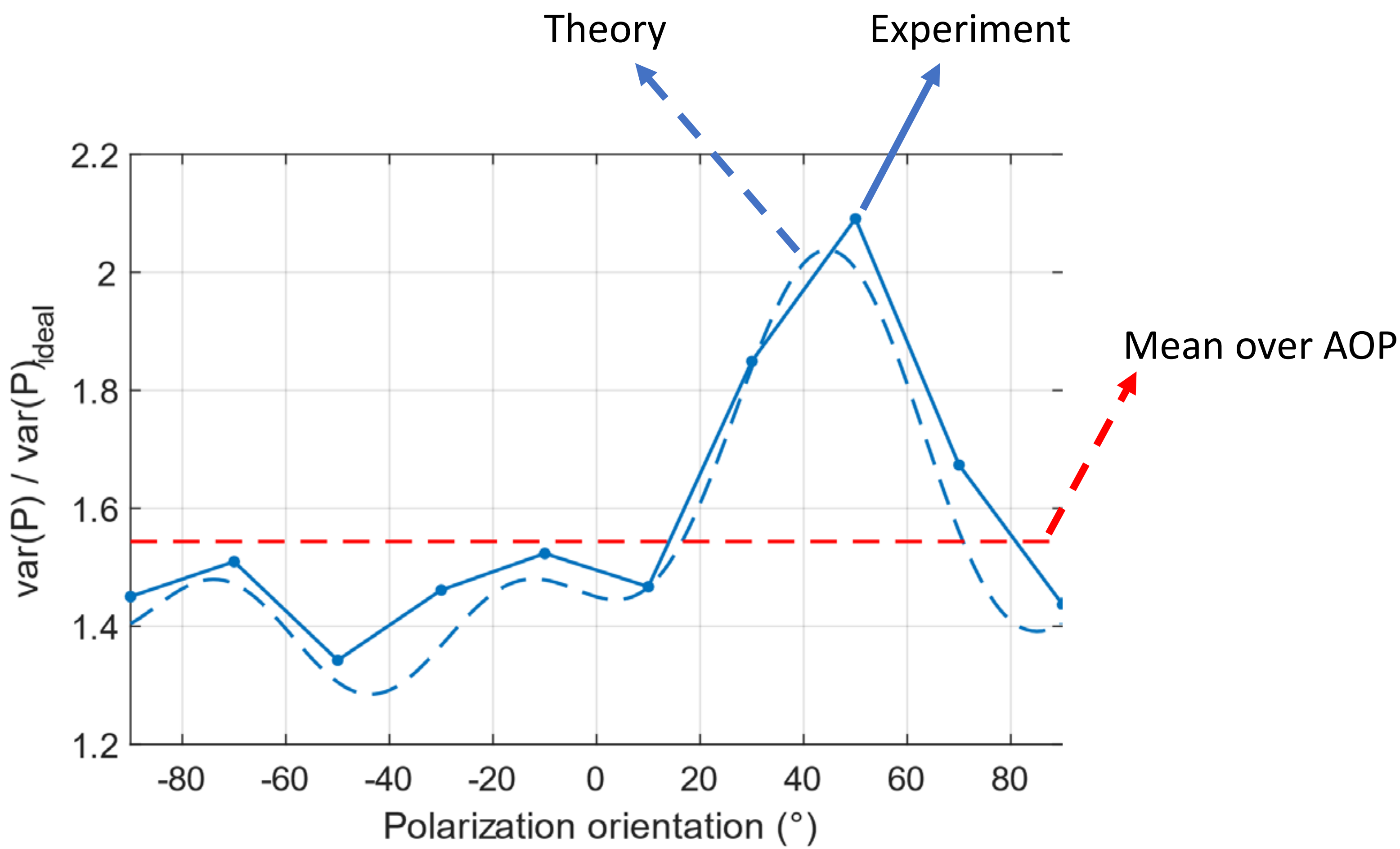
with

$c = \cos(2\alpha)$ et $s = \sin(2\alpha)$

$\delta_{ij} = g^2[(W^T W)^{-1}]_{ij}$ et $\gamma_{ij}^k = g\sum_{l=1}^4 W_{il}^\dagger W_{jl}^\dagger W_{lk}$, $\forall(k,i,j) \in [0,2]^3$

Roussel et al., Opt. Express (2018)

Variance of DOP



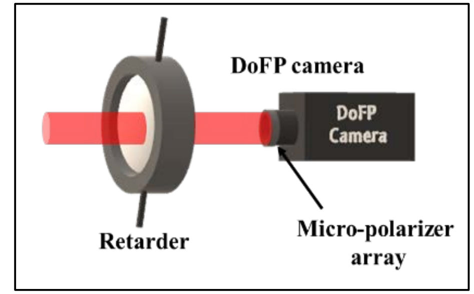
Roussel et al., Opt. Express (2018)

How to measure the full Stokes vector with DoFP camera ?

- Microgrid of retarders:

Hsu et al, « Full-Stokes imaging polarimeter using an array of elliptical polarizer, » *Optics Express* (2014)

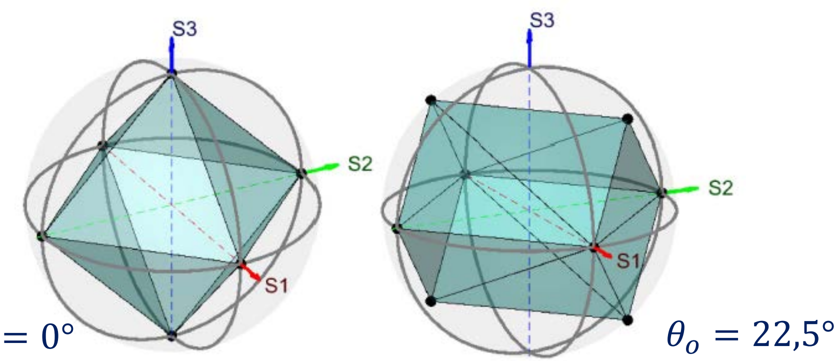
- Put a **retarder in front** of a linear DOFP camera



- One must perform **at least two acquisitions** :

➔ one obtains **8 intensity measurements** for each superpixel

Measurement vectors on **Poincaré** sphere



$\theta_o = 0°$ $\theta_o = 22{,}5°$

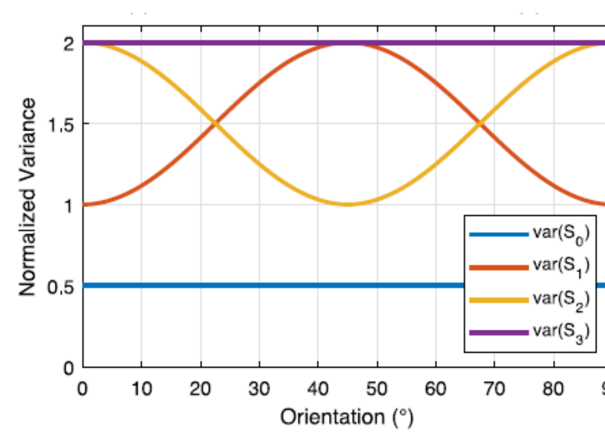
Roussel et al., Opt. Lett. (2019)

- Optimal **retarder parameters** (among others):
  - o Retardance : 90° (quarter wave plate)
  - o No retarder / Retarder with any angle θ_o

- Optimal **EWV : $5.5\sigma^2$**

  (to be compared to $^{40\sigma^2}/_{N} = 5\sigma^2$ for unconstrained polarimeter)

Variances of S_1 and S_2 **depend** on θ_o



Conclusion

- For estimating Stokes vector, the measurement matrices must correspond to **spherical 2 or 3 design**, depending on type of noise.

- One can predict the estimation precision of **polarimetric parameters** from linear Stokes vector **with optimal** or **imperfect** measurement matrices
  - These results have been recently generalized to parameters estimated from **full Stokes vector** (DOP, AOP, ellipticity)

    Dai et al., Opt. Express (2018) ***Dai et al., JOSA A (2019)***

- DoFPcameras can accelerate Stokes vector measurements: they **open up new exciting application** fields to polarization imaging (automotive navigation, ...)